Problem Set 11 – Relativity (G0I36A)

1 The Reissner-Nordström Black Hole Solution

1.) If we expand out the covariant derivatives in the field strength explicitly, we find that

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \left(\partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho\right) - \left(\partial_\nu A_\mu - \Gamma^\rho_{\nu\mu} A_\rho\right) = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{1.1}$$

where the last equality comes from noting that the Christoffel symbols all cancel due to their symmetry in the lower two indices. Thus, we can compute $F_{\mu\nu}$ simply by taking partial derivatives of the gauge field A_μ. Since only the A_t component is non-zero, and since A_t depends only on r, the only possible way to make $\partial_\mu A_\nu$ non-zero is to set $\mu = r$ and $\nu = t$. Thus, only $F_{tr} = -F_{rt}$ will be non-zero, and in particular

$$F_{tr} = \partial_t A_r - \partial_r A_t = -\partial_r A_t = -\partial_r \left(\frac{Q}{\sqrt{4\pi} r}\right) = \frac{Q}{\sqrt{4\pi} r^2} \ . \tag{1.2}$$

We could have also obtained the same result using form language. The gauge field A is a one-form, and the field strength $F = dA$ is a two-form given by evaluating the exterior derivative d on A. Explicitly, we compute

$$\begin{aligned} F &= dA \\ &= d\left(\frac{Q}{\sqrt{4\pi} r} dt\right) \\ &= \partial_\mu \left(\frac{Q}{\sqrt{4\pi} r}\right) dx^\mu \wedge dt \\ &= \partial_r \left(\frac{Q}{\sqrt{4\pi} r}\right) dr \wedge dt \\ &= -\frac{Q}{\sqrt{4\pi} r^2} dr \wedge dt \\ &= \frac{Q}{\sqrt{4\pi} r^2} dt \wedge dr \ . \end{aligned} \tag{1.3}$$

Thus we arrive at precisely the same result for the field strength.

2.) Now that we have F_{tr}, it's easy to compute that

$$F^{tr} = g^{tt} g^{rr} F_{tr} = -F_{tr} = -\frac{Q}{\sqrt{4\pi} r^2} \ , \tag{1.4}$$

since $g^{tt} = -\Delta^{-1}$ and $g^{rr} = \Delta$, as the metric is diagonal and so inverting it amounts to taking the inverse of each diagonal component. Our goal now is to show that this satisfies $\nabla_\mu F^{\mu\nu}$. To do so, we first write out this equation more explicitly as

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} \ , \tag{1.5}$$

using the definition of the covariant derivative. Notice that the last term must vanish identically, since $\Gamma^\nu_{\mu\rho}$ is symmetric under $\mu \leftrightarrow \rho$ while $F^{\mu\rho}$ is antisymmetric under the same exchange. Thus, we are left with

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} \ . \tag{1.6}$$

At this point we could simply compute all the relevant Christoffel symbols and verify that the Maxwell equation (1.6) vanishes for $\nu = t, r, \theta, \phi$; this approach will certainly work. However, I want to use a slightly more interesting method of proving this. Recall from problem set 8, question 2, that we proved the following identity:

$$\Gamma^{\nu}_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} \; . \tag{1.7}$$

Plugging this identity into the Maxwell equation (1.6), we thus find that

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}}\left(\partial_\rho \sqrt{-g}\right) F^{\rho\nu} \; . \tag{1.8}$$

Relabelling the ρ dummy index to μ and combining terms, we thus find that

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, F^{\mu\nu}\right) \; . \tag{1.9}$$

Therefore, in order to prove that $\nabla_\mu F^{\mu\nu}$ vanishes, it suffices to show simply that $\partial_\mu\left(\sqrt{-g}\, F^{\mu\nu}\right)$ is zero! Note that this is not true for general two-index tensors; antisymmetry was required to simplify the equation greatly, which led us to (1.6).

So, we can now prove that $\partial_\mu\left(\sqrt{-g}\, F^{\mu\nu}\right) = 0$ in order to prove that the Maxwell equations hold. First, we note that $\sqrt{-g} = r^2 \sin\theta$, which depends only on r and θ, while $F^{tr} = -F^{rt}$ depends only on r, and all other components of $F^{\mu\nu}$ are zero. Thus, the terms in this equation that involve ∂_t or ∂_ϕ vanish identically, and we are left with

$$\partial_\mu\left(\sqrt{-g}\, F^{\mu\nu}\right) = \partial_r\left(r^2 \sin\theta F^{r\nu}\right) + \partial_\theta\left(r^2 \sin\theta F^{\theta\nu}\right) \; . \tag{1.10}$$

The second term must be zero, since $F^{\theta\mu} = 0$ for all values of μ, so this reduces further to

$$\partial_\mu\left(\sqrt{-g}\, F^{\mu\nu}\right) = \partial_r\left(r^2 \sin\theta F^{r\nu}\right) \; . \tag{1.11}$$

The only possibly non-zero component of this equation is the $\nu = t$ component, since $F^{rt} \neq 0$. If we plug in our result, though, then this becomes

$$\partial_\mu\left(\sqrt{-g}\, F^{\mu t}\right) = \partial_r\left(r^2 \sin\theta \times \frac{Q}{\sqrt{4\pi} r^2}\right) = \partial_r\left(\frac{Q\sin\theta}{\sqrt{4\pi}}\right) = 0 \; . \tag{1.12}$$

So, all components of the Maxwell equation vanish.

3.) **Bonus:** if you are interested in seeing this done explicitly because you're having trouble with any step, e-mail me and I'll send you some notes. Typing it all up here would be a long exercise in LaTeX tedium.

4.) Since we can easily compute that $\sqrt{-g} = r^2 \sin\theta$, we therefore have that

$$\Box\psi = \frac{1}{r^2\sin\theta}\partial_\mu\left(r^2 \sin\theta\, g^{\mu\nu}\partial_\nu\psi\right) \; . \tag{1.13}$$

Then, since the metric is diagonal, this will boil down to just four individual terms:

$$\begin{aligned}\Box\psi = {} & \frac{1}{r^2\sin\theta}\partial_t\left(r^2\sin\theta\left(-\Delta^{-1}\right)\partial_t\psi\right) + \frac{1}{r^2\sin\theta}\partial_r\left(r^2\sin\theta\left(\Delta\right)\partial_r\psi\right) \\ & + \frac{1}{r^2\sin\theta}\partial_\theta\left(r^2\sin\theta\left(\frac{1}{r^2}\right)\partial_\theta\psi\right) + \frac{1}{r^2\sin\theta}\partial_\phi\left(r^2\sin\theta\left(\frac{1}{r^2\sin^2\theta}\right)\partial_\phi\psi\right) \; ,\end{aligned} \tag{1.14}$$

where we used that $g^{tt} = -\Delta^{-1}$, $g^{rr} = \Delta$, $g^{\theta\theta} = 1/r^2$, and $g^{\phi\phi} = 1/(r^2 \sin^2\theta)$. We can simplify these four terms immensely by pulling out all terms that are not affected by certain partial derivatives, taking special care to note that Δ depends on r but no other coordinate, after which point we are left with

$$\Box\psi = -\frac{1}{\Delta}\partial_t^2\psi + \frac{1}{r^2}\partial_r\left(r^2\Delta\partial_r\psi\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\psi\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi\ . \tag{1.15}$$

At this point it's not really possible to simplify the Laplacian anymore. Thus, we find that the Klein-Gordon equation $(\Box - m^2)\psi = 0$ is therefore given by

$$\boxed{0 = -\frac{1}{\Delta}\partial_t^2\psi + \frac{1}{r^2}\partial_r\left(r^2\Delta\partial_r\psi\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\psi\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi - m^2\psi}\ . \tag{1.16}$$

5.) Using the definition of $\hat{L}^2$, the Laplacian on S^2, we can more compactly write the Klein-Gordon equation above as

$$0 = -\frac{1}{\Delta}\partial_t^2\psi + \frac{1}{r^2}\partial_r\left(r^2\Delta\partial_r\psi\right) - \frac{1}{r^2}\hat{L}^2\psi - m^2\psi\ . \tag{1.17}$$

Since $\hat{L}^2$ acts directly on ψ without mixing with any other derivatives, and the t and r derivatives also show up in separate terms from one another, a very natural separation of variables ansatz is

$$\psi = T(t)R(r)Y_{\ell m}(\theta,\phi)\ , \tag{1.18}$$

where T and R are some functions of t and r, respectively, and $Y_{\ell m}(\theta,\phi)$ are spherical harmonics on S^2. Since they are eigenfunctions of $\hat{L}^2$ with eigenvalue $\ell(\ell+1)$, and $\hat{L}^2$ involves only θ and ϕ, we find that

$$\hat{L}^2\psi = \ell(\ell+1)\psi\ . \tag{1.19}$$

So, on our ansatz, the Klein-Gordon equation becomes

$$\begin{aligned} 0 = &-\frac{1}{\Delta}T''(t)R(r)Y_{\ell m}(\theta,\phi) + \frac{1}{r^2}\partial_r\left(r^2\Delta R'(r)\right)T(t)Y_{\ell m}(\theta,\phi) \\ &- \frac{\ell(\ell+1)}{r^2}T(t)R(r)Y_{\ell m}(\theta,\phi) - m^2T(t)R(r)Y_{\ell m}(\theta,\phi)\ . \end{aligned} \tag{1.20}$$

Note that $Y_{\ell m}(\theta,\psi)$ appears in all terms with no derivatives acting on it, and so we can simply cancel it out as a common factor in all terms. That is, we have chosen an ansatz such that all θ,ψ dependence of the Klein-Gordon equation drop out, leaving us with only t and r dependence:

$$0 = -\frac{1}{\Delta}T''(t)R(r) + \frac{1}{r^2}\partial_r\left(r^2\Delta R'(r)\right)T(t) - \frac{\ell(\ell+1)}{r^2}T(t)R(r) - m^2T(t)R(r)\ . \tag{1.21}$$

We would like to get rid of all t-dependence as well. The obstacle here is the $T''(t)$ term. We need to ansatz a particular form for $T(t)$ that is an eigenfunction of this second derivative, and so we will guess

$$T(t) = e^{iEt}\ , \tag{1.22}$$

such that $T''(t) = -E^2T(t)$ for some constant E. This simplifies the Klein-Gordon equation to

$$0 = -\frac{E^2}{\Delta}T(t)R(r) + \frac{1}{r^2}\partial_r\left(r^2\Delta R'(r)\right)T(t) - \frac{\ell(\ell+1)}{r^2}T(t)R(r) - m^2T(t)R(r)\ . \tag{1.23}$$

$T(t)$ now can be cancelled out from all terms, leaving us with simply a differential equation for $R(r)$:

$$0 = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\Delta(r)\frac{\partial}{\partial r}\right) - \frac{E^2}{\Delta(r)} - \frac{\ell(\ell+1)}{r^2} - m^2\right] R(r)\ . \tag{1.24}$$

This depends only on the radial coordinate r and the constants E, ℓ, and m. Thus, we have shown explicitly that there exists an ansatz of the form

$$\psi(t, r, \theta, \phi) = e^{iEt} R(r) Y_{\ell m}(\theta, \phi) \tag{1.25}$$

that demonstrates the separability of the Klein-Gordon equation on the Reissner-Nordström background. If we can solve the radial differential equation (1.24), then we can construct the most general solution to the full Klein-Gordon equation by taking superpositions of our ansatz for different values of E, ℓ, and m.

2 One Fish, Two Fish, Redshift, Blueshift

6.) Using the usual Reissner-Nordström coordinate frame given in the problem, there is some time coordinate separation Δt between Alice sending the photon off and Bob actually receiving the photon. However, this coordinate time separation is not the same as the actual proper time that Alice and Bob themselves feel. For any observer sitting at fixed radial coordinates and angular coordinates, i.e. an observer for which $dr = d\theta = d\phi = 0$, and the relation between their proper time coordinate τ and the coordinate time t is simply

$$ds^2 = -d\tau^2 = -\Delta(r)dt^2\ , \tag{2.1}$$

which in turn tells us that the proper time separation $\Delta\tau$ relates to Δt by

$$\Delta\tau = \sqrt{\Delta(r)}\Delta t\ . \tag{2.2}$$

Alice and Bob both sit at different radial coordinates r_1 and r_2, so they will not experience the same proper time, but the coordinate time between the photon leaving and Alice and getting to Bob is unambiguous. Thus, we have

$$\Delta\tau_1 = \sqrt{\Delta(r_1)}\,\Delta t\ , \quad \Delta\tau_2 = \sqrt{\Delta(r_2)}\,\Delta t\ , \tag{2.3}$$

and thus the ratio of the proper times is simply

$$\frac{\Delta\tau_1}{\Delta\tau_2} = \sqrt{\frac{\Delta(r_1)}{\Delta(r_2)}}\ . \tag{2.4}$$

Frequency is a measure of counts per second (as measured by each observer in their own inertial frame), and so the relative ratio f_1/f_2 of frequency in Alice's frame to frequency in Bob's frame will be inversely proportional to the ratio τ_1/τ_2 of proper times. Therefore we find that

$$\frac{f_2}{f_1} = \sqrt{\frac{\Delta(r_1)}{\Delta(r_2)}}\ . \tag{2.5}$$

Expanding the function Δ explicitly and rearranging terms slightly, we find the desired relationship:

$$f_2 = f_1\,\frac{r_2}{r_1}\sqrt{\frac{r_1^2 - 2GMr_1 + GQ^2}{r_2^2 - 2GMr_2 + GQ^2}}\ . \tag{2.6}$$

7.) The function $\Delta(r)$ is positive-definite anywhere outside the outer horizon r_+ of the black hole, and moreover it increases monotonically as we go from $r = r_+$ to $r = \infty$. So, if Alice is further from Bob, then $r_1 > r_2$, and so $\Delta(r_1)/\Delta(r_2) > 1$, and thus we conclude from (2.5) that $f_2 > f_1$. So, the frequency increases as it approaches Bob, and is thus blueshifted. If Bob is further out, the opposite is true, and the photon is redshifted. These conclusions are true irrespective of what values M and Q take, as long as we demand the extremality bound $Q^2 \leq GM^2$ is satisfied. (This bound guarantees that Δ increases monotonically outside the outer horizon). So, there are in fact no subtleties present in the Schwarzschild and extremal limits, making our lives easy.

8.) At the horizon, $\Delta(r) = 0$, and so the relation (2.5) tells us that the frequency Bob measures f_2 will blow up as he gets closer and closer to the horizon. That is, if Bob is sitting right at the horizon, the photon coming from Alice gets infinitely blueshifted as it approaches Bob. Since the quantum mechanical energy of the photon is $E = \hbar f$, this also means that the photon appears to be infinitely energetic from Bob's perspective.

Now, as soon as you see that an energy is infinite, you should always stop to think if this is okay or not. This is one of those few cases in which it's okay to have an infinite energy. This is because, even if the photon when it gets to Bob is infinitely energetic, it would require an infinitely large amount of energy to extract this energy and then escape the horizon of the black hole. The closer you are to the horizon, the harder it becomes to escape. So, there is sort of a conservation of difficulty here. This means that, even if the photon is infinitely more energetic at Bob as compared to how it was for Alice, we cannot easily extract this energy to do useful work unless we also put in a large amount of work into pulling the photon away from the black hole. So, we cannot use gravitational blueshift as a method to generate infinite renewable energy, unfortunately.